Complex Number

Roots of unity and Factorization

1. If $1, \omega, \omega^2$ are the cube roots of unity, $\omega^3 = 1$, $1+\omega+\omega^2 = \dfrac{1-\omega^3}{1-\omega} = \dfrac{0}{1-\omega} = 0$.

(i) $(a + \omega - \omega^2)(a - \omega + \omega^2) = a^2 - (\omega - \omega^2)^2 = a^2 - (\omega^2 - 2\omega^3 + \omega^4)$

$= a^2 - (\omega^2 - 2 + \omega)$, since $\omega^3 = 1$

$= a^2 - (1 + \omega + \omega^2 - 3) = a^2 - (0 - 3) = a^2 + 3$.

(ii) $(1 + i\omega - \omega^2)(1 - \omega + i\omega^2) = [1 + i\omega + (1 + \omega)][1 - \omega - i(1 + \omega)]$

$= [(2 + \omega) + i\omega][(1 - \omega) - i(1 + \omega)] = [(2 + \omega)(1 - \omega) + \omega(1 + \omega)] + i[\omega(1 - \omega) - (2 + \omega)(1 + \omega)]$

$= 2 - 2(1 + \omega + \omega^2) i = 2 + 0i = 2$

(iii) $(a + b)(a + b\omega)(a + b\omega^2) = a^3 + a^2b(1 + \omega + \omega^2) + ab^2(1 + \omega + \omega^2) + b^3\omega^3 = a^3 + b^3$

(iv) $(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= (a + b + c)[a^2 + b^2\omega^3 + c^2\omega^3 + ab(\omega + \omega^2) + bc(\omega^2 + \omega^4) + ca(\omega + \omega^2)]$

$= (a + b + c)[a^2 + b^2 + c^2 + ab(\omega + \omega^2) + bc(\omega + \omega^2) + ca(\omega + \omega^2)]$

$= (a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca] = a^3 + b^3 + c^3 - 3abc$

2. **(i)** $1 + \omega^r + \omega^{2r} = \dfrac{1-(\omega^r)^3}{1-\omega} = \dfrac{1-(\omega^3)^r}{1-\omega} = \dfrac{1-1^r}{1-\omega} = \dfrac{0}{1-\omega} = 0$, where r is not a multiple of 3 .

If r is a multiple of 3, $1 + \omega^r + \omega^{2r} = 1 + 1 + 1 = 3$.

(ii) $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) = x^2 + y^2 + z^2 - xy - yz - zx$, study 1(iv) for the expansion.

(iii) Let $f(x, y, z) = (x - y)^n + (y - z)^n + (z - x)^n$.

$f(-\omega y - \omega^2 z, y, z) = [(-\omega y - \omega^2 z) - y]^n + (y - z)^n + [z - (-\omega y - \omega^2 z)]^n$

$= [(-\omega - 1)y - \omega^2 z]^n + (y - z)^n + [\omega y + (1 + \omega^2)z]^n$

$= [\omega^2 y - \omega^2 z]^n + (y - z)^n + [\omega y + (-\omega)y]^n$, since $1 + \omega + \omega^2 = 0$

$= \omega^{2n}(y - z)^n + (y - z)^n + \omega^n (y - z)^n = (1 + \omega^n + \omega^{2n})(y - z)^n = 0\times(y - z)^n = 0$

By Factor theorem $(x + \omega y + \omega^2 z)$ is a factor of $f(x, y, z)$.

Similarly, $f(-\omega^2 y - \omega z, y, z) = 0$ and $(x + \omega^2 y + \omega z)$ is a factor of $f(x, y, z)$.

By (ii), $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) = x^2 + y^2 + z^2 - xy - yz - zx$ and result follows.

3. For $(x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b) \equiv x^3 - 3abx + a^3 + b^3$, same as 1.(iv) where $c = x$.

For the equation $x^3 - px + q = 0$, put $p = 3ab$ and $q = a^3 + b^3$,

we have $x^3 - 3abx + a^3 + b^3 = 0 \Leftrightarrow (x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b) = 0$

$\Leftrightarrow x = -(a + b), -(\omega a + \omega^2 b), -(\omega^2 a + \omega b)$ …. (1)

It remains to write a, b in terms of p and q.

$$\begin{cases} p = 3ab \\ q = a^3 + b^3 \end{cases} \Leftrightarrow \begin{cases} \text{Sum of roots} = a^3 + b^3 = q \\ \text{Product of roots} = (a^3)(b^3) = p^3/27 \end{cases} \Leftrightarrow a^3, b^3 \text{ are roots of } t^2 - qt + \frac{p^3}{27} = 0$$

$$\Leftrightarrow a^3, b^3 = \frac{1}{2}\left(q \pm \sqrt{q^2 - \frac{4p^3}{27}}\right) \Leftrightarrow a, b = \sqrt[3]{\frac{1}{2}\left(q \pm \sqrt{q^2 - \frac{4p^3}{27}}\right)} \qquad \text{…. (2)}$$

Substitute (2) in (1) will give the roots of the given cubic equation.

4.(a) $(1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$ …. (1)

Put $x = 1$ in (1), $(1 + 1)^n = c_0 + c_1 + c_2 + \dots + c_n$ …. (2)

Put $x = \omega$ in (1), $(1 + \omega)^n = c_0 + c_1\,\omega + c_2\,\omega^2 + \dots + c_n\,\omega^n$ …. (3)

Put $x = \omega^2$ in (1), $(1 + \omega^2)^n = c_0 + c_1\,\omega^2 + c_2\,\omega^4 + \dots + c_n\,\omega^{2n}$ …. (4)

Adding (2), (3) and (4) and noting that $1 + \omega^r + \omega^{2r} = 0$, if r is not a multiple of 3,

$\therefore\quad 2^n + (1 + \omega)^n + (1 + \omega^2)^n = 3(c_0 + c_3 + \dots)$ …. (5)

Now, $1+\omega = 1+\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} = 2\cos\frac{\pi}{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = \cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$\therefore\quad (1 + \omega)^n = \cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}$, by de Moivre's Theorem .

Also, $(1 + \omega^2)^n = \cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}$. Hence, from (5), $c_0 + c_3 + \dots = \frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$

(b) $(c_0+c_3+\dots)^2 = \frac{1}{9}\left(2^n+2\cos\frac{n\pi}{3}\right)^2 = \frac{1}{9}\left(4^n+2^{n+2}\cos\frac{n\pi}{3}+4\cos^2\frac{n\pi}{3}\right)$ …. (6)

Multiply (3) by ω^2, (4) by ω and add these to (1), we get

$2^n + \omega^2(1 + \omega)^n + \omega(1 + \omega^2)^n = 3(c_1 + c_4 + \dots)$ …. (7)

Now, $\omega^2(1+\omega)^n = \left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right) = \cos\frac{(n-2)\pi}{3}+i\sin\frac{(n-2)\pi}{3}$

and $\omega\,(1+\omega^2)^n = \cos\frac{(n-2)\pi}{3}-i\sin\frac{(n-2)\pi}{3}$. From (7), $c_1 + c_4 + \dots = \frac{1}{3}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right)$

and $(c_1+c_4+\dots)^2 = \frac{1}{9}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right)^2 = \frac{1}{9}\left(4^n+2^{n+2}\cos\frac{(n-2)\pi}{3}+4\cos^2\frac{(n-2)\pi}{3}\right)$ …. (8)

Finally, multiply (3) by ω, (4) by ω^2 and add these to (1), we get

$2^n + \omega(1 + \omega)^n + \omega^2(1 + \omega^2)^n = 3(c_2 + c_5 + \dots)$ …. (9)

Now, $\omega(1+\omega^2)^n = \left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right) = \cos\frac{(n+2)\pi}{3}+i\sin\frac{(n+2)\pi}{3}$

Similarly, $\omega^2(1+\omega)^n = \cos\frac{(n+2)\pi}{3}-i\sin\frac{(n+2)\pi}{3}$

Hence, from (9), $c_2 + c_5 + \dots = \frac{1}{3}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right)$

and $(c_2+c_5+\dots)^2 = \frac{1}{9}\left(2^n+2\cos\frac{(n+2)\pi}{3}\right)^2 = \frac{1}{9}\left(4^n+2^{n+2}\cos\frac{(n+2)\pi}{3}+4\cos^2\frac{(n+2)\pi}{3}\right)$ …. (10)

(6) + (8) + (10), $(c_0+c_3+\dots)^2+(c_1+c_4+\dots)^2+(c_2+c_5+\dots)^2$

$= \frac{1}{3}4^n+\frac{2^{n+2}}{9}\left(\cos\frac{(n-2)\pi}{3}+\cos\frac{n\pi}{3}+\cos\frac{(n+2)\pi}{3}\right)+\frac{4}{9}\left(\cos^2\frac{(n-2)\pi}{3}+\cos^2\frac{n\pi}{3}+\cos^2\frac{(n+2)\pi}{3}\right)$ …. (11)

But $\cos\frac{(n-2)\pi}{3}+\cos\frac{n\pi}{3}+\cos\frac{(n+2)\pi}{3} = \mathrm{Re}\left[\mathrm{cis}\frac{(n-2)\pi}{3}+\mathrm{cis}\frac{n\pi}{3}+\mathrm{cis}\frac{(n+2)\pi}{3}\right]$

$= \mathrm{Re}\left[\mathrm{cis}\frac{(n-2)\pi}{3}\left(1+\mathrm{cis}\frac{2\pi}{3}+\mathrm{cis}\frac{4\pi}{3}\right)\right] = \mathrm{Re}\left[\mathrm{cis}\frac{(n-2)\pi}{3}\left(1+\omega+\omega^2\right)\right] = \mathrm{Re}(0) = 0$ …. (12)

and $\cos^2\frac{(n-2)\pi}{3}+\cos^2\frac{n\pi}{3}+\cos^2\frac{(n+2)\pi}{3} = \frac{1}{2}\left[\left(1+\cos\frac{2(n-2)\pi}{3}\right)+\left(1+\cos\frac{2n\pi}{3}\right)+\left(1+\cos\frac{2(n+2)\pi}{3}\right)\right]$

$= \frac{3}{2}+\mathrm{Re}\left[\mathrm{cis}\frac{2(n-2)\pi}{3}+\mathrm{cis}\frac{2n\pi}{3}+\mathrm{cis}\frac{2(n+2)\pi}{3}\right] = \frac{3}{2}+\mathrm{Re}\left[\mathrm{cis}\frac{2(n-2)\pi}{3}\left(1+\omega^2+\omega^4\right)\right] = \frac{3}{2}+0 = \frac{3}{2}$ …. (13)

(12), (13) ↓(11), $(c_0+c_3+\dots)^2+(c_1+c_4+\dots)^2+(c_2+c_5+\dots)^2 = \frac{1}{3}4^n+\frac{2^{n+2}}{9}(0)+\frac{4}{9}\left(\frac{3}{2}\right) = \frac{1}{3}\left(4^n+2\right)$.

5. **(i)** Consider the equation $f(x) = x^8 - 4x^4 + 16 = (x^4)^2 - 4(x^4) + 16 = 0$

$$x^4 = \frac{4 \pm \sqrt{4^2 - 4(16)}}{2} = 2 \pm 2\sqrt{3}i = 4\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) = 8\text{cis}\left(\pm\frac{\pi}{3}\right) = 8\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$$

$$\therefore \quad x = \left[8\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)\right]^{1/4} = \sqrt{2}\left(\cos\frac{6n\pi+\pi}{12} \pm i\sin\frac{6n\pi+\pi}{12}\right) \quad \text{, where } n = 0, 1, 2, 3.$$

For each conjugate roots, by taking $\theta = \dfrac{6n\pi+\pi}{12}$ we form factor of the form :

$$\left[x - \sqrt{2}(\cos\theta + i\sin\theta)\right]\left[x - \sqrt{2}(\cos\theta - i\sin\theta)\right] = \left(x - \sqrt{2}\cos\theta\right)^2 - \left(\sqrt{2}i\sin\theta\right)^2$$

$$= x^2 - \left(2\sqrt{2}\cos\theta\right)x + 2\cos^2\theta + 2\sin^2\theta = x^2 - \left(2\sqrt{2}\cos\theta\right)x + 2 = x^2 - \left(2\sqrt{2}\cos\frac{6n\pi+\pi}{12}\right)x + 2$$

$\therefore \quad x^8 - 4x^4 + 16$

$$= \left[x^2 - \left(2\sqrt{2}\cos\frac{\pi}{12}\right)x + 2\right]\left[x^2 - \left(2\sqrt{2}\cos\frac{7\pi}{12}\right)x + 2\right]\left[x^2 - \left(2\sqrt{2}\cos\frac{13\pi}{12}\right)x + 2\right]\left[x^2 - \left(2\sqrt{2}\cos\frac{19\pi}{12}\right)x + 2\right]$$

$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1 + \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{1}{8}\left(4 + 2\sqrt{3}\right) = \frac{1}{8}\left(\sqrt{3}+1\right)^2 \Rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Similarly, $\cos\dfrac{7\pi}{12} = -\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\quad \cos\dfrac{13\pi}{12} = -\dfrac{\sqrt{3}+1}{2\sqrt{2}}$, $\quad \cos\dfrac{19\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$$\therefore \quad x^8 - 4x^4 + 16 = \left[x^2 - \left(\sqrt{3}+1\right)x + 2\right]\left[x^2 + \left(\sqrt{3}-1\right)x + 2\right]\left[x^2 + \left(\sqrt{3}+1\right)x + 2\right]\left[x^2 - \left(\sqrt{3}-1\right)x + 2\right]$$

(ii) Consider the equation : $f(x) = x^6 + 8x^3 + 64 = (x^3)^2 + 8(x^3) + 64 = 0$,

$$x^3 = \frac{-8 \pm \sqrt{8^2 - 4(64)}}{2} = -4 \pm 4\sqrt{3}i = 8\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) = 8\text{cis}\left(\pm\frac{2\pi}{3}\right) = 8\left[\cos\left(\frac{2\pi}{3}\right) \pm i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$x = 2\left[\cos\left(\frac{2\pi}{3}\right) \pm i\sin\left(\frac{2\pi}{3}\right)\right]^{1/3} = 2\left[\cos\left(\frac{6n\pi+2\pi}{9}\right) \pm i\sin\left(\frac{6n\pi+2\pi}{9}\right)\right] \quad \text{, where } n = 0, 1, 2.$$

For each conjugate roots, by taking $\theta = \dfrac{6n\pi+2\pi}{9} = \dfrac{(3n+1)2\pi}{9}$ we form factor of the form :

$$\left[x - 2(\cos\theta + i\sin\theta)\right]\left[x - 2(\cos\theta - i\sin\theta)\right] = (x - 2\cos\theta)^2 - (2i\sin\theta)^2$$

$$= x^2 - (4\cos\theta)x + 4\cos^2\theta + 4\sin^2\theta = x^2 - (4\cos\theta)x + 4 = x^2 - \left(4\cos\frac{(3n+1)2\pi}{9}\right)x + 4$$

$$\therefore \quad x^6 + 8x^3 + 64 = \left[x^2 - \left(4\cos\frac{2\pi}{9}\right)x + 4\right]\left[x^2 - \left(4\cos\frac{8\pi}{9}\right)x + 4\right]\left[x^2 - \left(4\cos\frac{14\pi}{9}\right)x + 4\right]$$

6. Consider $f(x) = x^9 - 1 = 0 \quad \Rightarrow x = \text{cis}\left(\dfrac{2n\pi}{9}\right) \quad n = 0, 1, \ldots, 8$

$$\therefore \quad x = 1, \quad \text{cis}\left(\pm\frac{2\pi}{9}\right), \quad \text{cis}\left(\pm\frac{4\pi}{9}\right), \quad \text{cis}\left(\pm\frac{6\pi}{9}\right), \quad \text{cis}\left(\pm\frac{8\pi}{9}\right) .$$

Combining factors with conjugate roots as in No. 5, we have :

$$x^9 - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{6\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{8\pi}{9}x + 1\right) \quad \ldots. \quad (1)$$

(i) $g(x) = x^3 - 1 = 0 \quad \Rightarrow x = \text{cis}\left(\frac{2n\pi}{3}\right) \quad n = 0, 1, 2. \quad \Rightarrow \quad x = 1, \quad \pm\text{cis}\frac{2\pi}{3}$

Combining factors with conjugate roots as in No. 5, we have :

$$x^3 - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{3}x + 1\right) \qquad \ldots. \quad (2)$$

From (1), since $x^9 - 1 = (x^3 - 1)(x^6 + x^3 + 1)$ and use (2) to cancel the appropriate factors,

$$\therefore \quad x^6 + x^3 + 1 = \left(x^2 - 2\cos\frac{2\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{8\pi}{9}x + 1\right) \qquad \ldots. \quad (3)$$

Divide L.H.S. of (3) by x^3, and each factor of R.H.S. of (3) by x, we have

$$x^3 + \frac{1}{x^3} + 1 = \left(x + \frac{1}{x} - 2\cos\frac{2\pi}{9}\right)\left(x + \frac{1}{x} - 2\cos\frac{4\pi}{9}\right)\left(x + \frac{1}{x} - 2\cos\frac{8\pi}{9}\right) \qquad \ldots. \quad (4)$$

(ii) Put $x = 1$ in (4),

$$3 = \left(2 - 2\cos\frac{2\pi}{9}\right)\left(2 - 2\cos\frac{4\pi}{9}\right)\left(2 - 2\cos\frac{8\pi}{9}\right) = 8\left(1 - \cos\frac{2\pi}{9}\right)\left(1 - \cos\frac{4\pi}{9}\right)\left(1 - \cos\frac{8\pi}{9}\right)$$

$$\Rightarrow \frac{3}{8} = \left(2\sin^2\frac{\pi}{9}\right)\left(2\sin^2\frac{2\pi}{9}\right)\left(2\sin^2\frac{4\pi}{9}\right) \Rightarrow \frac{3}{64} = \left(\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}\right)^2 \Rightarrow \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9} = \frac{\sqrt{3}}{8}$$

(iii) $x = \cos\theta + i\sin\theta, \ \frac{1}{x} = \cos\theta - i\sin\theta \Rightarrow x^3 = \cos 3\theta + i\sin 3\theta, \ \frac{1}{x^3} = \cos 3\theta - i\sin 3\theta$

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta, \ x^3 + \frac{1}{x^3} = 2\cos 3\theta$$

From (4), $\quad 2\cos 3\theta + 1 = \left(2\cos\theta - 2\cos\frac{2\pi}{9}\right)\left(2\cos\theta - 2\cos\frac{4\pi}{9}\right)\left(2\cos\theta - 2\cos\frac{8\pi}{9}\right)$

$$2\cos 3\theta + 1 = 8\left(\cos\theta - \cos\frac{2\pi}{9}\right)\left(\cos\theta - \cos\frac{4\pi}{9}\right)\left(\cos\theta - \cos\frac{8\pi}{9}\right) \qquad \ldots. \quad (5)$$

Replace θ by $\pi + \theta$ in (5),

$$2\cos 3(\pi+\theta) + 1 = 8\left(\cos(\pi+\theta) - \cos\frac{2\pi}{9}\right)\left(\cos(\pi+\theta) - \cos\frac{4\pi}{9}\right)\left(\cos(\pi+\theta) - \cos\frac{8\pi}{9}\right)$$

$$-2\cos 3\theta + 1 = 8\left(-\cos\theta - \cos\frac{2\pi}{9}\right)\left(-\cos\theta - \cos\frac{4\pi}{9}\right)\left(-\cos\theta - \cos\frac{8\pi}{9}\right)$$

$$2\cos 3\theta - 1 = 8\left(\cos\theta + \cos\frac{2\pi}{9}\right)\left(\cos\theta + \cos\frac{4\pi}{9}\right)\left(\cos\theta + \cos\frac{8\pi}{9}\right) \qquad \ldots. \quad (6)$$

(5)×(6), $\quad 64\left(\cos^2\theta - \cos^2\frac{2\pi}{9}\right)\left(\cos^2\theta - \cos^2\frac{4\pi}{9}\right)\left(\cos^2\theta - \cos^2\frac{8\pi}{9}\right) = 4\cos^2 3\theta - 1$

Since $\cos^2\frac{8\pi}{9} = \cos^2\frac{\pi}{9}$, $\quad 64\left(\cos^2\theta - \cos^2\frac{\pi}{9}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{9}\right)\left(\cos^2\theta - \cos^2\frac{4\pi}{9}\right) = 4\cos^2 3\theta - 1$

7. **(i)** Consider the equation $x^{2n} - 1 = 0$, the roots are

$$x = (\text{cis } 0)^{1/2n} = (\text{cis } 2k\pi)^{1/2n} = \text{cis}\frac{2k\pi}{2n}, \quad k = 0, 1, \ldots, 2n-1.$$

$$= 1, -1 \ \text{ or } \ \text{cis}\left(\pm\frac{k\pi}{n}\right), \qquad k = 1, \ldots, n-1.$$

$$\therefore \quad x^{2n} - 1 = (x-1)(x+1)\prod_{k=1}^{n-1}\left[x - \text{cis}\frac{k\pi}{n}\right]\left[x - \text{cis}\left(-\frac{k\pi}{n}\right)\right]$$

(ii) $\dfrac{1}{x^{2n}-1}=\sum_{k=1}^{2n}\dfrac{A_k}{x-\alpha_k}$. By Partial fraction theorem, $\dfrac{P(x)}{Q(x)}=\sum_{k=1}^{2n}\dfrac{P(\alpha_k)}{Q'(\alpha_k)(x-\alpha_k)}$

where $Q(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{2n})$

$P(x) = 1 \Rightarrow P(\alpha_k) = 1$. $Q'(x) = 2n\,x^{2n-1} \Rightarrow Q'(\alpha_k) = 2n\,\alpha_k^{\,2n-1}$

If α_k is a root of $x^{2n} - 1 = 0$, then the partial fraction corresponding to the factor $x-\alpha_k$ is

$\dfrac{1}{2n\alpha_k^{\,2n-1}(x-\alpha_k)}$ .Taking the two fractions corresponding to the conjugate values of

α_k , $\alpha_{-k}=\cos\dfrac{k\pi}{n}\pm i\sin\dfrac{k\pi}{n}$ together, we get the fraction :

$$\frac{1}{2n\alpha_k^{\,2n-1}(x-\alpha_k)}+\frac{1}{2n\alpha_{-k}^{\,2n-1}(x-\alpha_{-k})}=\frac{1}{2n}\frac{\alpha_{-k}^{\,2n-1}(x-\alpha_k)+\alpha_k^{\,2n-1}(x-\alpha_{-k})}{\alpha_k^{\,2n-1}\alpha_{-k}^{\,2n-1}(x-\alpha_k)(x-\alpha_{-k})}$$

$$=\frac{1}{2n}\frac{\left(\alpha_k^{\,2n-1}+\alpha_{-k}^{\,2n-1}\right)x-\left(\alpha_{-k}\alpha_k^{\,2n-1}+\alpha_k\alpha_{-k}^{\,2n-1}\right)}{(x-\alpha_k)(x-\alpha_{-k})}=\frac{1}{2n}\frac{\left(\alpha_k^{\,2n-1}+\alpha_{-k}^{\,2n-1}\right)x-\left(\alpha_k^{\,2n-2}+\alpha_{-k}^{\,2n-2}\right)}{x^2-(\alpha_k+\alpha_{-k})x+\alpha_k\alpha_{-k}}$$

$$=\frac{1}{2n}\frac{2\left(\cos\frac{k\pi}{n}(2n-1)\right)x^2-2\left(\cos\frac{k\pi}{n}(2n-2)\right)x}{x^2-2\left(\cos\frac{k\pi}{n}(2n-1)\right)x+1}=\frac{1}{n}\frac{\left(\cos\frac{k\pi}{n}(2n-1)\right)x^2-\left(\cos\frac{k\pi}{n}(2n-2)\right)x}{x^2-2\left(\cos\frac{k\pi}{n}(2n-1)\right)x+1}$$

$$\therefore\quad \frac{1}{x^{2n}-1}=\frac{1}{2n(x-1)}-\frac{1}{2n(x+1)}+\sum_{k=1}^{n-1}\frac{1}{n}\frac{\left(\cos\frac{k\pi}{n}(2n-1)\right)x^2-\left(\cos\frac{k\pi}{n}(2n-2)\right)x}{x^2-2\left(\cos\frac{k\pi}{n}(2n-1)\right)x+1}$$

8. $F(z) = (z+1)^8 - z^8 = 0 \quad \Rightarrow (z+1)^8 = z^8 \Rightarrow \left(\dfrac{z+1}{z}\right)^8 = 1 \Rightarrow \dfrac{z+1}{z}=(\text{cis }2k\pi)^{1/8}=\text{cis }\dfrac{k\pi}{4}$, k = 0, 1, … , 7

$z=\dfrac{1}{\text{cis }\frac{k\pi}{4}-1}$, k = 1, 2, …, 7 (When k = 0 , z = 1 is obviously not a root of the original equation.)

$\therefore$ The roots are **(1)** $z = -\dfrac{1}{2}$ (when k = 4) or

(2) $z=\dfrac{1}{\text{cis}\left(\pm\frac{k\pi}{4}\right)-1}$ k = 1, 2, 3.

$$=\frac{1}{\text{cis}\left(\pm\frac{k\pi}{8}\right)\left(\text{cis}\left(\pm\frac{k\pi}{8}\right)-\text{cis}\left(\mp\frac{k\pi}{8}\right)\right)}=\frac{1}{\text{cis}\left(\pm\frac{k\pi}{8}\right)\left(2i\sin\left(\pm\frac{k\pi}{8}\right)\right)}=\frac{-i}{2\sin\left(\pm\frac{k\pi}{8}\right)}\left(\cos\left(\pm\frac{k\pi}{8}\right)-i\sin\left(\pm\frac{k\pi}{8}\right)\right)$$

$$=-\frac{1}{2}-\frac{1}{2}\cot\left(\pm\frac{k\pi}{8}\right)-i\sin\left(\cos\left(\pm\frac{k\pi}{8}\right)-i\sin\left(\pm\frac{k\pi}{8}\right)\right)=-\frac{1}{2}-\frac{1}{2}\cot\left(\pm\frac{k\pi}{8}\right)-i\sin\frac{\theta}{2}=-\frac{1}{2}-\frac{1}{2}\cot\left(\pm\frac{k\pi}{8}\right)i \quad \ldots(1)$$

For corresponding factors of f(z),

(1) (2z + 1) is a factor of f(z) corresponding to the root $z = -\dfrac{1}{2}$ for the equation f(z) = 0 .

(2) From roots in (1), we can form 3 real quadratic factors where k = 1, 2, 3:

$$\left[z+\frac{1}{2}+\frac{1}{2}\cot\left(\frac{k\pi}{8}\right)i\right]\left[z+\frac{1}{2}-\frac{1}{2}\cot\left(\frac{k\pi}{8}\right)i\right]=\left(z+\frac{1}{2}\right)^2-\left[\frac{1}{2}\cot\left(\frac{k\pi}{8}\right)i\right]^2=z^2+z+\frac{1}{4}\csc^2\frac{k\pi}{8}$$

$$\therefore\quad (z+1)^8-z^8=A(2z+1)\prod_{k=1}^{3}\left\{z^2+z+\frac{1}{4}\csc^2\frac{k\pi}{8}\right\}=\frac{A}{8}(2z+1)\prod_{k=1}^{3}\left\{4z^2+4z+\csc^2\frac{k\pi}{8}\right\}$$

By comparing coeff. of z^7, we then get A = $\frac{1}{2}$, $\therefore\ (z+1)^8 - z^8 = \frac{1}{16}(2z+1)\prod_{k=1}^{3}\left\{4z^2+4z+\csc^2\frac{k\pi}{8}\right\}$

Put $z = \cos^2\theta - 1 = -\sin^2\theta$, $z + 1 = \cos^2\theta$, $2z + 1 = 2(\cos^2\theta - 1) + 1 = \cos 2\theta$

$4z^2 + 4z = (2z + 1)^2 - 1 = \cos^2 2\theta - 1$, we get

$$\left(\cos^2\theta\right)^8 - \left(-\sin^2\theta\right)^8 = \frac{1}{16}\cos 2\theta\ \prod_{k=1}^{3}\left\{\cos^2 2\theta - 1 + \csc^2\frac{k\pi}{8}\right\}$$

$$\therefore\quad 16\left(\cos^{16}\theta - \sin^{16}\theta\right) = \cos 2\theta\ \prod_{k=1}^{3}\left\{\cos^2 2\theta + \cot^2\frac{k\pi}{8}\right\}$$

9. $(1 + x)^{2n+1} = (1 - x)^{2n+1} \quad \Leftrightarrow \left(\frac{1+x}{1-x}\right)^{2n+1} = 1 \Leftrightarrow \frac{1+x}{1-x} = \operatorname{cis}\frac{2r\pi}{2n+1}$, where $r = 0, 1, \ldots, 2n$.

$\Leftrightarrow \frac{1+x}{1-x} = \operatorname{cis}\left(\pm\frac{r\pi}{2n+1}\right)$, where $r = 1, \ldots, n$. ($r = 0$ case is rejected for non-zero root)

$$x = \frac{\operatorname{cis}\left(\pm\frac{2r\pi}{2n+1}\right)-1}{\operatorname{cis}\left(\pm\frac{2r\pi}{2n+1}\right)+1} = \frac{\operatorname{cis}\left(\pm\frac{r\pi}{2n+1}\right)\left[\operatorname{cis}\left(\pm\frac{r\pi}{2n+1}\right)-\operatorname{cis}\left(\mp\frac{r\pi}{2n+1}\right)\right]}{\operatorname{cis}\left(\pm\frac{r\pi}{2n+1}\right)\left[\operatorname{cis}\left(\pm\frac{r\pi}{2n+1}\right)+\operatorname{cis}\left(\mp\frac{r\pi}{2n+1}\right)\right]} = \frac{2i\sin\left(\pm\frac{r\pi}{2n+1}\right)}{2\cos\left(\pm\frac{r\pi}{2n+1}\right)} = \pm i\tan\left(\frac{r\pi}{2n+1}\right) \text{ ..(1)}$$

When $n = 2$, the equation is $(1 + x)^5 - (1 - x)^5 = 0$ or $x(x^4 + 10x^2 + 5) = 0$

For the non-zero root case, we consider the equation : $x^4 + 10x^2 + 5 = 0$

The product of roots = coeff of constant term / coeff of x^4 term = 5 and from (1)

$\therefore \left(i\tan\frac{\pi}{5}\right)\times\left(i\tan\frac{2\pi}{5}\right)\times\left(-i\tan\frac{\pi}{5}\right)\times\left(-i\tan\frac{2\pi}{5}\right) = 5 \quad \Rightarrow \quad \tan^2\frac{\pi}{5}\times\tan^2\frac{2\pi}{5} = 5$.

10. $(z + 1)^{2n} + (z - 1)^{2n} = 0 \quad \Leftrightarrow \left(\frac{z+1}{z-1}\right)^{2n} = -1 = \operatorname{cis}\pi \Leftrightarrow \frac{z+1}{z-1} = (\operatorname{cis}\pi)^{1/2n} = \operatorname{cis}\frac{\pi+2k\pi}{2n} = \operatorname{cis}\theta$

where $k = 0, 1, .., (2n - 1)$.

$$z = \frac{\operatorname{cis}\theta+1}{\operatorname{cis}\theta-1} = \frac{\operatorname{cis}(\theta/2)}{\operatorname{cis}(\theta/2)}\frac{\operatorname{cis}(\theta/2)+\operatorname{cis}\left[-(\theta/2)\right]}{\operatorname{cis}(\theta/2)-\operatorname{cis}\left[-(\theta/2)\right]} = \frac{2\cos(\theta/2)}{2i\sin(\theta/2)} = -i\cot(\theta/2) = -i\cot\left(\frac{\pi+2k\pi}{n}\right) = i\cot\left(\frac{\pi+2k\pi}{n}\right)$$

which is purely imaginary.

$(z + 1)^{2n} + (z - 1)^{2n} = 0 \quad \Leftrightarrow 2z^{2n} + 0\times z^{2n-1} + 2n(2n - 1)\, z^{2n-2} + \ldots + 2 = 0$

$\Leftrightarrow z^{2n} + 0\times z^{2n-1} + n(2n - 1)\, z^{2n-2} + \ldots + 1 = 0$

Sum of roots = $\sum_{k=0}^{2n-1} i\cot\left(\frac{\pi+2k\pi}{n}\right) = 0$,

Sum of pairs of roots = $\sum_{k,r=0,k\neq r}^{2n-1}\left[i\cot\left(\frac{\pi+2k\pi}{n}\right)\right]\left[i\cot\left(\frac{\pi+2r\pi}{n}\right)\right] = n(2n-1)$

$OP_1^{\,2} + OP_2^{\,2} + \ldots + OP_{2n}^{\,2} = \sum_{k=0}^{2n-1}\left[\cot\left(\frac{\pi+2k\pi}{n}\right)\right]^2 = -\sum_{k=0}^{2n-1}\left[i\cot\left(\frac{\pi+2k\pi}{n}\right)\right]^2$

$$= -\left[\sum_{k=0}^{2n-1} i\cot\left(\frac{\pi+2k\pi}{n}\right)\right]^2 + 2\sum_{k,r=0,k\neq r}^{2n-1}\left[i\cot\left(\frac{\pi+2k\pi}{n}\right)\right]\left[i\cot\left(\frac{\pi+2r\pi}{n}\right)\right]$$

$= -0^2 + 2\,[n(2n - 1)] = \quad 2n(2n - 1)$.

11. **(i)** Following the discussion in 7(i) , taking $\theta = \frac{r\pi}{n}$ we combine conjugate factors :

$$[x - (\cos\theta + i\sin\theta)][x - (\cos\theta - i\sin\theta)] = (x - \cos\theta)^2 - (i\sin\theta)^2$$

$$= x^2 - (2\cos\theta)x + \cos^2\theta + \sin^2\theta = x^2 - (2\cos\theta)x + 1 = x^2 - \left(2\cos\frac{r\pi}{n}\right)x + 1$$

$$\therefore \quad x^{2n} - 1 = (x-1)(x+1)\prod_{r=1}^{n-1}\left[x^2 - \left(2\cos\frac{r\pi}{n}\right)x + 1\right] \quad \ldots. \quad (1)$$

(ii) Consider the equation $x^{2n+1} - 1 = 0$, the roots are

$$x = (\text{cis } 0)^{\frac{1}{2n+1}} = (\text{cis } 2k\pi)^{\frac{1}{2n+1}} = \text{cis}\frac{2r\pi}{2n+1} \quad , \quad r = 0, 1, \ldots, 2n .$$

$$= 1 \text{ or } \text{cis}\left(\pm\frac{2r\pi}{2n+1}\right), \qquad r = 1, 2, \ldots, n .$$

$$\therefore \quad x^{2n+1} - 1 = (x-1)\prod_{r=1}^{n}\left[x - \text{cis}\frac{2r\pi}{2n+1}\right]\left[x - \text{cis}\left(-\frac{2r\pi}{2n+1}\right)\right] = (x-1)\prod_{r=1}^{n}\left(x^2 - 2x\cos\frac{2r\pi}{2n+1} + 1\right) \quad \ldots. \quad (2)$$

If $x \neq 0$, from (1), $x^n(x^n - x^{-n}) = [x(x - x^{-1})]\prod_{r=1}^{n-1}\left[x(x + x^{-1}) - x\left(2\cos\frac{r\pi}{n}\right)\right]$ divide both sides of by x^n ,

we have $$x^n - x^{-n} = (x - x^{-1})\prod_{r=1}^{n-1}\left(x + x^{-1} - 2\cos\frac{r\pi}{n}\right) \quad \ldots. \quad (3)$$

Divide (3) by $x - x^{-1}$, $x^{n-1} + x^{n-3} + x^{n-5} + \ldots + x^{-(n-3)} + x^{-(n-1)} = \prod_{r=1}^{n-1}\left(x + x^{-1} - 2\cos\frac{r\pi}{n}\right)$

Put $x = 1$, $n = \prod_{r=1}^{n-1}\left(2 - 2\cos\frac{r\pi}{n}\right) \Rightarrow n = 2^{n-1}\prod_{r=1}^{n-1}\left(1 - \cos\frac{r\pi}{n}\right) \Rightarrow n = 2^{n-1}\prod_{r=1}^{n-1}2\left(\sin^2\frac{r\pi}{2n}\right) = 2^{2(n-1)}\left[\prod_{r=1}^{n-1}\left(\sin\frac{r\pi}{2n}\right)\right]^2$

$$\Rightarrow \frac{n}{2^{2(n-1)}} = \left[\prod_{r=1}^{n-1}\left(\sin\frac{r\pi}{2n}\right)\right]^2 \Rightarrow \prod_{r=1}^{n-1}\sin\frac{r\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$$

12. Consider the equation : $x^{2n} - 2x^n\cos n\theta + 1 = 0 \Rightarrow \quad (x^n)^2 - (2\cos n\theta)(x^n) + 1 = 0$

$$x^n = \cos n\theta \pm \sqrt{\cos^2 n\theta - 1} = \cos n\theta \pm i\sin n\theta \Rightarrow x = (\cos n\theta \pm i\sin n\theta)^{1/n} = \cos\frac{n\theta + 2k\pi}{n} \pm i\sin\frac{n\theta + 2k\pi}{n}$$

$\therefore \quad x = \cos\left(\theta + \frac{2k\pi}{n}\right) \pm i\sin\left(\theta + \frac{2k\pi}{n}\right)$ where $k = 0, 1, \ldots, (n-1)$

The corresponding conjugate factors :

$$\left\{x - \left[\cos\left(\theta + \frac{2k\pi}{n}\right) + i\sin\left(\theta + \frac{2k\pi}{n}\right)\right]\right\}\left\{x - \left[\cos\left(\theta + \frac{2k\pi}{n}\right) - i\sin\left(\theta + \frac{2k\pi}{n}\right)\right]\right\} = x^2 - 2x\cos\left(\theta + \frac{2k\pi}{n}\right) + 1$$

$$\therefore \quad x^{2n} - 2x^n\cos n\theta + 1 = \prod_{k=0}^{n-1}\left[x^2 - 2x\cos\left(\theta + \frac{2k\pi}{n}\right) + 1\right] \quad \ldots. \quad (1)$$

(i) Put $x = 1$, $\theta = 2\alpha$ in (1),

$$2 - 2\cos 2n\alpha = \prod_{k=0}^{n-1}\left[2 - 2\cos\left(2\alpha + \frac{2k\pi}{n}\right)\right] \Rightarrow 4\sin^2 n\alpha = \prod_{k=0}^{n-1}\left[4\sin^2\left(\alpha + \frac{k\pi}{n}\right)\right]$$

$$\Rightarrow 2^2\sin^2 n\alpha = 2^{2n}\left[\prod_{k=0}^{n-1}\sin^2\left(\alpha + \frac{k\pi}{n}\right)\right]^2 \Rightarrow \sin n\alpha = 2^{n-1}\prod_{k=0}^{n-1}\sin\left(\alpha + \frac{k\pi}{n}\right) \quad \ldots. \quad (2)$$

(ii) From (2), Replace α by $\dfrac{\beta - \alpha}{2}$ and α by $\dfrac{\beta+\alpha}{2}$ we get

$$\sin\frac{n\beta - n\alpha}{2} = 2^{n-1}\prod_{k=0}^{n-1}\sin\left(\frac{\beta-\alpha}{2}+\frac{k\pi}{n}\right) \quad \text{and} \quad \sin\frac{n\beta+n\alpha}{2} = 2^{n-1}\prod_{k=0}^{n-1}\sin\left(\frac{\beta+\alpha}{2}+\frac{k\pi}{n}\right)$$

Multiply these two identities and use compound angle formula to change it to a difference,

$$\frac{1}{2}(\cos n\alpha - \cos n\beta) = 2^{2(n-1)}\prod_{k=0}^{n-1}\frac{1}{2}\left[\cos\alpha - \cos\left(\beta + \frac{2k\pi}{n}\right)\right] = 2^{2(n-1)}\frac{1}{2^n}\prod_{k=0}^{n-1}\left[\cos\alpha - \cos\left(\beta+\frac{2k\pi}{n}\right)\right]$$

$$\therefore \quad \cos n\alpha - \cos n\beta = 2^{n-1}\prod_{k=0}^{n-1}\left[\cos\alpha - \cos\left(\beta+\frac{2k\pi}{n}\right)\right] \qquad \ldots.\ (3)$$

(iii) Take logarithm of (1), $\ln(\sin n\alpha) = \ln(2^{n-1}) + \sum_{k=0}^{n-1}\ln\left(\sin\left(\alpha+\frac{k\pi}{n}\right)\right)$ …. (4)

Differentiate (4) w.r.t. α, $\dfrac{n}{\sin n\alpha} = \sum_{k=0}^{n-1}\left[1\Big/\sin\left(\alpha+\frac{k\pi}{n}\right)\right]$ …. (5)

$$\therefore \quad \cot n\alpha = \frac{1}{n}\sum_{k=0}^{n-1}\cot\left(\alpha+\frac{k\pi}{n}\right), \qquad \alpha \neq \frac{r\pi}{n}$$

(iv) Differentiate (5), $-n\csc^2 n\alpha = \dfrac{1}{n}\sum_{k=0}^{n-1}\left[-\csc^2\left(\alpha+\frac{k\pi}{n}\right)\right] \Rightarrow \csc^2 n\alpha = \dfrac{1}{n^2}\sum_{k=0}^{n-1}\csc^2\left(\alpha+\frac{k\pi}{n}\right), \quad \alpha \neq \dfrac{r\pi}{n}$.

13. **(i)** Consider the equation : $x^{2n} - 2x^n a^n\cos n\theta + a^{2n} = 0 \Rightarrow (x^n)^2 - (2a^n\cos n\theta)(x^n) + a^{2n} = 0$

$$x^n = a^n\cos n\theta \pm \sqrt{a^{2n}\cos^2 n\theta - a^{2n}} = a^n(\cos n\theta \pm i\sin n\theta) \Rightarrow x = a\left[\cos\frac{n\theta+2k\pi}{n} \pm i\sin\frac{n\theta+2k\pi}{n}\right]$$

$$x = a\left[\cos\left(\theta+\frac{2k\pi}{n}\right) \pm i\sin\left(\theta+\frac{2k\pi}{n}\right)\right] \text{ where } k = 0, 1, \ldots, (n-1)$$

The corresponding conjugate factors :

$$\left\{x - a\left[\cos\left(\theta+\frac{2k\pi}{n}\right)+i\sin\left(\theta+\frac{2k\pi}{n}\right)\right]\right\}\left\{x - a\left[\cos\left(\theta+\frac{2k\pi}{n}\right)-i\sin\left(\theta+\frac{2k\pi}{n}\right)\right]\right\} = x^2 - 2xa\cos\left(\theta+\frac{2k\pi}{n}\right)+a^2$$

$$\therefore \quad x^{2n} - 2x^n a^n\cos n\theta + a^{2n} = \prod_{k=0}^{n-1}\left[x^2 - 2xa\cos\left(\theta+\frac{2k\pi}{n}\right)+a^2\right] \qquad \ldots.\ (1)$$

(ii) By taking $x = a = 1$, $\theta = 2\alpha$ in (1),

$$2 - 2\cos 2n\alpha = \prod_{k=0}^{n-1}\left[2 - 2\cos\left(2\alpha+\frac{2k\pi}{n}\right)\right] \Rightarrow \sin n\alpha = 2^{n-1}\prod_{k=0}^{n-1}\sin\left(\alpha+\frac{k\pi}{n}\right) \qquad \ldots.\ (2)$$

By taking $x = a = 1$, $\theta = 2\alpha - \pi$ in (1),

$$\sin n\left(\alpha - \frac{\pi}{2}\right) = 2^{n-1}\prod_{k=0}^{n-1}\sin\left(\alpha - \frac{\pi}{2}+\frac{k\pi}{n}\right) = (-1)^n 2^{n-1}\prod_{k=0}^{n-1}\cos\left(\alpha+\frac{k\pi}{n}\right) \qquad \ldots.\ (3)$$

$$\therefore \quad E = \prod_{r=0}^{n-1}\sin^2\left(\alpha+\frac{r\pi}{n}\right)+\prod_{r=0}^{n-1}\cos^2\left(\alpha+\frac{r\pi}{n}\right) = \frac{\sin^2 n\alpha}{2^{2n-2}}+\frac{1}{2^{2n-2}}\sin^2 n\left(\alpha-\frac{\pi}{2}\right)$$

When n is odd, $E = \dfrac{\sin^2 n\alpha}{2^{2n-2}} + \dfrac{\cos^2 n\alpha}{2^{2n-2}} = \dfrac{1}{2^{2n-2}} = 2^{2-2n}$

When n is even, $E = \dfrac{\sin^2 n\alpha}{2^{2n-2}} + \dfrac{\sin^2 n\alpha}{2^{2n-2}} = \dfrac{2}{2^{2n-2}}\sin^2 n\alpha = 2^{3-2n}\sin^2 n\alpha$

(iii) Take logarithm of (1)

$$\ln\left(x^{2n} - 2x^n a^n \cos n\theta + a^{2n}\right) = \sum_{k=0}^{n-1} \ln\left[x^2 - 2xa\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2\right]$$

Differentiate and divide by 2, $\dfrac{nx^{n-1}\left(x^n - a^n\cos n\theta\right)}{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}} = \sum_{r=0}^{n-1} \dfrac{x - a\cos\left(\theta + \frac{2r\pi}{n}\right)}{x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2}$ …. (4)

(iv) Differentiate L.H.S. of (4),

$$\frac{\left(x^{2n} - 2x^n a^n \cos n\theta + a^{2n}\right)\left[(2n-1)nx^{2n-2} - n(n-1)a^n x^{n-2}\cos n\theta\right] - 2\left[nx^{n-1}\left(x^n - a^n\cos n\theta\right)\right]^2}{\left(x^{2n} - 2x^n a^n \cos n\theta + a^{2n}\right)^2}$$

Put $x = a = 1$,

$$\frac{(2 - 2\cos n\theta)\left[(2n-1)n - n(n-1)\cos n\theta\right] - 2\left[n(1-\cos n\theta)\right]^2}{(2 - 2\cos n\theta)^2} = \frac{\left[(2n-1)n - n(n-1)\cos n\theta\right] - n^2(1-\cos n\theta)}{2 - 2\cos n\theta}$$

$$= \frac{n^2 - n(1-\cos n\theta)}{2(1-\cos n\theta)} = \frac{n^2}{2(1-\cos n\theta)} - \frac{n}{2} \quad \text{…. (5)}$$

Differentiate R.H.S. of (4),

$$\sum_{r=0}^{n-1} \frac{\left[x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right] - 2\left[x - a\cos\left(\theta + \frac{2r\pi}{n}\right)\right]^2}{\left[x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right]^2}$$

Put $x = a = 1$, $\displaystyle\sum_{r=0}^{n-1} \frac{\left[2 - 2\cos\left(\theta + \frac{2r\pi}{n}\right)\right] - 2\left[1 - \cos\left(\theta + \frac{2r\pi}{n}\right)\right]^2}{\left[2 - 2\cos\left(\theta + \frac{2r\pi}{n}\right)\right]^2}$

$$= \sum_{r=0}^{n-1} \frac{1 - \left[1 - \cos\left(\theta + \frac{2r\pi}{n}\right)\right]}{2 - 2\cos\left(\theta + \frac{2r\pi}{n}\right)} = \frac{1}{2}\sum_{r=0}^{n-1} \frac{1}{1 - \cos\left(\theta + \frac{2r\pi}{n}\right)} - \frac{n}{2} \quad \text{…. (6)}$$

Equate (5) and (6), $\displaystyle\sum_{r=0}^{n-1} \frac{1}{1 - \cos\left(\theta + \frac{2r\pi}{n}\right)} = \frac{n^2}{1 - \cos n\theta}$.

(v) In **(i)** , take $x = i$, $a = 1$ and n even, (subscript k is replaced by r to avoid ambiguity)

$$1 - 2(-1)^{n/2}\cos n\theta + 1 = \prod_{r=0}^{n-1}\left[-1 - 2i\cos\left(\theta + \frac{2r\pi}{n}\right) + 1\right] \Rightarrow 2 - 2(-1)^{n/2}\cos n\theta = 2^n(-1)^n(-1)^{n/2}\prod_{r=0}^{n-1}\left[\cos\left(\theta + \frac{2r\pi}{n}\right)\right]$$

Put $n = 2k$, $\displaystyle 2 - 2(-1)^k\cos 2k\theta = 2^{2k}(-1)^{2k}(-1)^k\prod_{r=0}^{2k-1}\left[\cos\left(\theta + \frac{2r\pi}{2k}\right)\right]$,

$$\therefore \quad 2^{2k-1}\cos\theta\cos\left(\theta + \frac{\pi}{k}\right)\cos\left(\theta + \frac{2\pi}{k}\right) \ldots \cos\left(\theta + \frac{(2k-1)\pi}{k}\right) = (-1)^k - \cos 2k\theta \quad \text{…. (7)}$$

Replace θ by $\left(\frac{\pi}{2} + \theta\right)$ in (7) and note that $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$,

$$2^{2k-1}(-1)^{2k}\sin\theta\sin\left(\theta + \frac{\pi}{k}\right)\sin\left(\theta + \frac{2\pi}{k}\right) \ldots \sin\left(\theta + \frac{(2k-1)\pi}{k}\right) = (-1)^k - (-1)^k\cos 2k\theta$$

$$\therefore \quad 2^{2k-1}\sin\theta\sin\left(\theta+\frac{\pi}{k}\right)\sin\left(\theta+\frac{2\pi}{k}\right)\ \ldots\ \sin\left(\theta+\frac{(2k-1)\pi}{k}\right)=(-1)^k(1-\cos 2k\theta) \qquad \ldots. \quad (8)$$

(vi) Put $k = n$ in (8),

$$2^{2n-1}\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ \ldots\ \sin\left(\theta+\frac{(2n-1)\pi}{n}\right)=(-1)^n(1-\cos 2n\theta) \qquad \ldots. \quad (9)$$

Replace θ by $\frac{\pi}{4n}$ in (9),

$$2^{2n-1}\sin\frac{\pi}{4n}\sin\left(\frac{\pi}{4n}+\frac{\pi}{n}\right)\sin\left(\frac{\pi}{4n}+\frac{2\pi}{n}\right)\ \ldots\ \sin\left(\frac{\pi}{4n}+\frac{(2n-1)\pi}{n}\right)=(-1)^n\left(1-\cos 2n\frac{\pi}{4n}\right)$$

$$2^{2n-1}\sin\frac{\pi}{4n}\sin\left(\frac{5\pi}{4n}\right)\sin\left(\frac{9\pi}{4n}\right)\ \ldots\sin\left(\frac{(4n-3)\pi}{4n}\right)\sin\left(\frac{(4n+1)\pi}{4n}\right)\ldots\sin\left(\frac{(8n-3)\pi}{4n}\right)=(-1)^n$$

$$2^{2n-1}\sin\frac{\pi}{4n}\sin\left(\frac{5\pi}{4n}\right)\sin\left(\frac{9\pi}{4n}\right)\ \ldots\sin\left(\frac{(4n-3)\pi}{4n}\right)\sin\left(\pi+\frac{\pi}{4n}\right)\ldots\sin\left(\pi+\frac{(4n-3)\pi}{4n}\right)=(-1)^n$$

$$2^{2n-1}(-1)^n\left[\sin\frac{\pi}{4n}\sin\left(\frac{5\pi}{4n}\right)\sin\left(\frac{9\pi}{4n}\right)\ \ldots\sin\left(\frac{(4n-3)\pi}{4n}\right)\right]^2=(-1)^n$$

$$\therefore \qquad \prod_{r=0}^{n-1}\sin\frac{(4r+1)\pi}{4n}=2^{\frac{1}{2}-n}$$

14. Each side of the polygon subtends an angle $\frac{2\pi}{n}$ at O.

Since $\angle POA_0 = \theta$, $\angle POA_k = \theta+\frac{2k\pi}{n}$, where $k = 0, 1, \ldots, (n-1)$.

By cosine law, $PA_k{}^2 = OP^2 - 2a(OP)\cos\left(\theta+\frac{2k\pi}{n}\right)+a^2 = x^2-2ax\cos\left(\theta+\frac{2k\pi}{n}\right)+a^2$

$$\prod_{k=0}^{n-1}PA_k{}^2=\prod_{k=0}^{n-1}\left[x^2-2ax\cos\left(\theta+\frac{2k\pi}{n}\right)+a^2\right]$$

By **13 (i)**, $x^{2n}-2x^na^n\cos n\theta+a^{2n}=\prod_{k=0}^{n-1}\left[x^2-2xa\cos\left(\theta+\frac{2k\pi}{n}\right)+a^2\right]$, therefore we have

$$\prod_{k=0}^{n-1}PA_k{}^2=x^{2n}-2x^na^n\cos n\theta+a^{2n}$$

$$\therefore \quad \prod_{k=0}^{n-1}PA_r=\sqrt{x^{2n}-2x^na^n\cos n\theta+a^{2n}}$$